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To	:	See MAS Distribution List No. 3
Subject	:	STANAG 4278 EL (EDITION 3) - METHOD OF EXPRESSING NAVIGATION ACCURACIES
References	:	a. MAS/93-EL/4278 dated 29 April 1986 (Edition 2) b. AC/302(SG/4)D/120 dated 17 August 1993 (Edition 3) (1st Draft)
Enclosure	:	STANAG 4278 (Edition 3)

1. The enclosed NATO Standardization Agreement which has been ratified by nations as reflected in page iii is promulgated herewith.

2. The references listed above are to be destroyed in accordance with local document destruction procedures.

3. AAP-4 should be amended to reflect the latest status of the STANAG.

ACTION BY NATIONAL STAFFS

4. National staffs are requested to examine page iii of the STANAG and if they have not already done so, to advise the Defence Support Division, IS, through their national delegation as appropriate of their intention regarding its ratification and implementation.

G.B. FERRARI
Major-General, ITAF
Chairman, MAS

STANAG 4278
(Edition 3)

NORTH ATLANTIC TREATY ORGANIZATION

(NATO)

MILITARY AGENCY FOR STANDARDIZATION

(MAS)

STANDARDIZATION AGREEMENT

SUBJECT: METHOD OF EXPRESSING NAVIGATION ACCURACIES

Promulgated on 7 June 1995

G.B. FERRARI
Major-General, ITAF
Chairman, MAS

STANAG 4278
(Edition 3)

RECORD OF AMENDMENTS

No.	Reference/date of amendment	Date entered	Signature

EXPLANATORY NOTES

AGREEMENT

1. This NATO Standardization Agreement (STANAG) is promulgated by the Chairman MAS under the authority vested in him by the NATO Military Committee.

2. No departure may be made from the agreement without consultation with the tasking authority. Nations may propose changes at any time to the tasking authority where they will be processed in the same manner as the original agreement.

3. Ratifying nations have agreed that national orders, manuals and instructions implementing this STANAG will include a reference to the STANAG number for purposes of identification.

DEFINITIONS

4. Ratification is "The declaration by which a nation formally accepts the content of this Standardization Agreement".

5. Implementation is "The fulfilment by a nation of its obligations under this Standardization Agreement".

6. Reservation is "The stated qualification by a nation which describes that part of this Standardization Agreement which it cannot implement or can implement only with limitations".

RATIFICATION, IMPLEMENTATION AND RESERVATIONS

7. Page iii gives the details of ratification and implementation of this agreement. If no details are shown it signifies that the nation has not yet notified the tasking authority of its intentions. Page iv (and subsequent) gives details of reservations and proprietary rights that have been stated.

STANAG 4278
(Edition 3)

NAVY/ARMY/AIR

NATO STANDARDIZATION AGREEMENT

(STANAG)

METHOD OF EXPRESSING NAVIGATION ACCURACIES

ANNEX : A. Method of Expressing Navigation Accuracies

AIM

1. The aims of this agreement are:

1.a To standardize the method of stating navigation accuracy in NATO operational and requirements documents.

1.b To ensure that new NATO technical documents which give statements of navigation performance include standard expressions of navigation accuracy, types of navigation accuracy, and conditions under which the accuracy is to be interpreted or measured.

AGREEMENT

2. Participating nations agree:

2.a Whenever navigation accuracies are discussed among NATO members in requirements or operational matters, the accuracies will be described in the following statistical terms:

(1) Linear (single dimension) accuracy. The measure of accuracy will be given as the distance in metres centred on the true value within which is contained 95% of observations.

(2) Two-dimensional accuracy. The measure of accuracy will be given as the radial distance in metres centred on the true position of a large number of trials which includes 95% of observations.

(3) Radio navigation aids. When it is desirable to express the performance of radio navigation systems, independent of geometrical factors, the measure will

represent the limit in appropriate dimensions (e.g. hertz, centicycles, microseconds, etc.) within which 95% of observations are contained.

(4) Velocity or speed and angles or angular rates. If performance statements include velocity, speed, angles or angular rates, the measure will represent the limit in appropriate dimensions (e.g. knots) within which 95% of observations are contained.

(5) Time and distance dependent navigation performance. For autonomous navigation systems such as Inertial and Doppler Navigation Systems, where errors accumulate over time or distance, the measure will be the distance or radius per unit of time or a percentage of distance travelled which contains 95% of observations (e.g. 1 knot or 1% of distance travelled). For aircraft Inertial Navigation Systems, accuracy will normally be represented into dimensions as the radius in speed in knots, distance in nautical miles as a function of mission time without update from external sources which contains 95% of observations.

(6) All statements of navigation accuracy will indicate the number of dimensions applicable to the statement.

(7) If the system performance under assessment has an error distribution whose mean is removed from zero, the statement will include the magnitude and direction of the bias.

(8) If the system performance under assessment has an error distribution which varies significantly from the Standard (or Gaussian) Distribution, the error distribution will be described to the extent possible.

(9) Input relationships. For systems whose accuracy is affected by the quality of input data (such as TRANSIT where an error in velocity input results in an error in estimated position) the relationships should be described (e.g., 0.1 nm position error for each 1 mps velocity input error) as well as the 95 percentile limit of the input source (e.g. 3 mps in the above example) so that the probable performance of the integrated system can be assessed.

(10) Precise time and time interval. Where precise time or time interval data are provided, the time or time interval accuracy will be expressed as a single dimensional number (microseconds, nanoseconds, etc.) encompassing 95% of the observations from a large number of trials.

2.b New and revised documentation treating navigation system accuracy will include the measures agreed to above.

2.c Whenever navigation accuracies are discussed among NATO members in requirements or operational matters, one or more of the following standardized types of accuracy shall be specified, as applicable:

(1) Geographic Accuracy. The accuracy of a position with respect to the geographic, or geodetic, coordinates of the earth.

(2) Repeatable Accuracy. The accuracy with which a user can return to a position whose system coordinates

have been measured at a previous time with the same navigation system.

(3) Relative Accuracy. The accuracy with which a user can measure position relative to that of another use of the same navigation system at the same time.

2.d Whenever navigation accuracies are discussed among NATO members in requirements or operational matters, the conditions under which the accuracy is to be interpreted or measured (e.g., platform dynamics, fix rate, time or place of applicability, etc.) shall be specified.

2.e Whenever navigation accuracies are discussed among NATO members in requirements or operational matters, the coordinate system and the time system with respect to which the navigation accuracy and time accuracy are respectively specified shall be specified. Whenever geographic accuracy is discussed, the earth model (geodetic spheroid) and associated datum reference (e.g., ED50, WGS 84 or NAD 1927) shall be specified.

2.f This agreement does not preclude the use of additional customary methods of expressing navigation performance but seeks to ensure that standardized measures are available for comparison and that misunderstandings caused by different terms of expression are avoided.

IMPLEMENTATION OF THE AGREEMENT

3. This agreement is considered implemented when national instructions have been issued to direct use of standard measures of accuracy in NATO documentation.

METHOD OF EXPRESSING NAVIGATION ACCURACIES

APPENDICES : A.1. Percentage Probability for Standard Error Increments

A.2. Bibliography

1. Summary

This Annex outlines recommended procedures and concepts to enhance the quality of information exchanged concerning navigation system accuracies. This enhancement may be realized by implementing consistency and clarity in the statement of statistics associated with navigation performance parameter characterization. Guidelines are provided for the implementation of STANAG 4278 and this Annex in no way supersedes that Agreement. A rigorous treatment of error theory or statistics is not intended; heuristic arguments primarily applicable to navigation considerations and the resolution of misconceptions/confusions regarding commonly used navigation measures are addressed.

1.1 Although specific data processing guidelines are stated, it is recognized that a rigid dependence on strict formulations may lead to false assurances and sad surprises. Notwithstanding the primary emphasis herein as the establishment of data processing guidelines for accuracy measures, the investigatory aspects of navigation error analysis, which include a) the matching of appropriate accuracy measures with the mission requirements b) the design of testing and observation procedures, c) the testing of assumptions regarding ergodicity, normality and independence, are critical.

2. Background

2.1 There are numerous methods of stating navigation system accuracies with various statistical characterizations. The method of analysis and assumptions made by evaluators may introduce inconsistencies among navigation accuracy measures despite using the same basic statistical technique. To enhance the information exchanged among NATO members it is desirable to identify which statistical method is used in accuracy performance statements, as well as associated confidence intervals.

2.2 This subject area has been discussed at length by AC/302(SG/4) and there is general agreement that an orderly method of presenting navigation accuracies should be standardized.

3. Discussion

3.1 Discussion of Errors.

3.1.1 The true error of a navigation quantity is the difference between the measured value and true value of the quantity. If the true quantity is known to be constant, then variations among successive measurements are the errors in the observations. Errors fall into three general classes which may be categorized by origin as (1) blunders, (2) systematic, and (3) random. Blunders are mistakes caused by misreading scales, transposing figures, erroneous computations, or careless observers. They are often large and usually detectable by repeated measurements. Systematic errors follow some fixed law and are generally constant in magnitude and/or sign within a series of observations under similar conditions. Systematic errors can often be eliminated or substantially reduced when the cause is known. An example of a systematic error in navigation might be the error incurred by not compensating for known atmospheric effects in the measurement of a range with a radio navigation system. Blunders and systematic errors should generally be separately described and quantified; the justification and methodology for their quantification should be given. Random errors are those remaining after blunders and systematic errors have been removed. They are the main source of errors treated in this Annex and are discussed in more detail below.

3.1.2 Random Errors

3.1.2.1 Random errors generally result from combinations of causes unknown and uncontrollable to the system under assessment. Accuracy measures, which are statistics, are used in navigation to quantitatively describe random errors. Random errors can further be divided into stationary and non-stationary errors. Stationary errors are errors whose statistics over a large number of trials are not functions of mission time or distance, or other deterministic mission parameter. Non-stationary random errors are errors whose statistics change with time or distance from some mission initialization point, or with some other deterministic mission parameter such as altitude of the flight path. Non-stationary errors in navigation are often associated with dead reckoning systems such as inertial navigation systems and Doppler radar navigators whose errors accumulate in time due to inherent integrations of random errors.

3.2 Navigation Accuracy Measures

3.2.1 Navigation accuracy measures are statistics which are used to quantitatively describe navigation random errors. For didactic purposes, navigation accuracy measures shall be divided into 1) Probability Indices and 2) Accuracy Formulas.

3.2.1.1 A Probability Index is defined based on the probability that the random errors lying within an interval (for one dimensional), an area (for two dimensional), a volume (for three dimensional), or a hypersurface (for multidimensional) has a specific value. For example the three dimensional probability index Spherical Error Probable (SEP) is the radius of the sphere defined such that the probability of an error lying within the sphere is 0.5. Error analysis, in which is embedded the assumption of an underlying population density distribution of the errors, allows one to relate Probability Indices to accuracy formulas (exact or approximate). For example, under the assumptions of a zero mean, uncorrelated, spherical (equal standard deviations in all three dimensions), jointly Gaussian three dimensional distribution, a formula would be SEP=1.5382 σ where σ is the standard deviation.

3.2.1.2 An Accuracy Formula is defined as a specific formula to compute a statistic which is used to infer navigation accuracy. For example the Mean Radial Spherical Error (MRSE) is defined as $\sqrt{\sigma_X^2+\sigma_Y^2+\sigma_Z^2}$ where σ_X, σ_Y, σ_Z are the error standard deviations in the x, y, and z directions respectively. We may utilize error analysis, which assumes a particular underlying population density distribution, to derive or approximate the probability that the error lies within the radius of a sphere given by the MRSE. For example, given the assumption of the same three dimensional probability distribution, we may derive that the probability is 0.6082 that the error lies within a sphere of radius MRSE (1.732σ).

3.2.1.2.1 It is noted from the above definitions that Accuracy Formulas and Probability Indices are inherently different types of accuracy measures. Each can be obtained from the data (random error sample) without reference to an underlying distribution. The probability indices can be determined directly from its definition as the number of samples being within the interval divided by the total number of samples. The accuracy formulas can be computed directly from application of their defining algorithm on the sample. Conversions (without going back to the actual data sample) among probability indices or between probability indices and accuracy formulas must assume an underlying population density function. Conversion between accuracy formulas using the same parameters can be done directly.

3.2.1.3 Once having obtained the values of the random sample it is usually desirable to use these sample values to make some inference(s) about the population represented by the sample which in the present context means the joint probability density distribution function of the random navigation errors for the system under assessment. Under some circumstances, the direct "histogramming" of data errors, which would yield a non-parametric distribution, may be the preferred manner of describing the errors. Although non-parametric methods (methods that do not use an underlying probability density distribution) are available, it is customary in navigation to assume (if applicable) Normal (Gaussian or Standard) distributions. The Normal distribution has the advantages of mathematical tractability relative to other distributions, conformity to the Central Limit Theorem, and a long tradition of usage in navigation. This is not an endorsement for its usage if the data does not support its applicability. Various goodness-of-fit tests, such as Kolmogorov-Smirnov and chi-square can be used to test for normality. Reports including navigation accuracy measures should address conformance of the data to the Normal distribution, and provide justification for the utilization of formulas.

3.2.2 Linear (One-dimensional) Stationary Errors

3.2.2.1 Scalar random error quantities in navigation such as range, time, speed, one-dimensional position, angle errors etc. apply to this section.

3.2.2.1.1 True errors (discounting blunders or systematic errors which are separately reported) are the differences between the actual measurements of the system under assessment and the reference truth. A discussion of the appropriateness of the reference truth is beyond the scope; however, if the error distribution of the reference truth is known, standard methods of determining the resultant error distribution should be utilized.

3.2.2.2. The true errors are represented as X_1, X_2, ..., X_n. The true errors will in general have a bias or sample mean value different from zero and defined as $\sum X_i / n$. For stationary errors, over a large number of independent tests with different random value sets this mean should approach a constant value. This mean value shall be reported separately.

3.2.2.3 The unbiased or residual errors shall be the differences between the true error and the mean value. When no reference truth is available and the navigation quantity under assessment is known to be a constant, then the mean of the random observations shall be used to construct unbiased errors. The unbiased errors shall be represented as $e_1, e_2, \ldots, e_n$.

3.2.2.4 The probability index accuracy measure for unbiased one-dimensional errors shall be the smallest l such that the probability of the absolute value of e less than or equal to l is greater than or equal to P (0<P<1). Or more concisely :

$$Prob(|e| \leq l) \geq P$$

The preferred value for P for NATO applications is 0.95 (95 percent probability index). It is noted that the probability index when applied to the unbiased errors represents the dispersion of the errors. The value for P of 0.5 (50% probability index) corresponds to the commonly used Linear Error Probable.

3.2.2.5 When the normality assumption is valid, the probability density functions for the unbiased errors are given by (See Figure 1):

$$P(e) = \frac{1}{\sigma_X \sqrt{2\pi}} \exp\left(-e^2 / 2\sigma_X^2\right)$$

$$\text{where } e_i = X_i - \sum X_i/n = X_i - \bar{X}$$

$$\text{and } \sigma_X = \sqrt{\frac{\sum (X_i - \bar{X})^2}{n-1}}$$

3.2.2.6 When the normality assumption is valid, the 95 percent (preferred for NATO) probability index is calculated as $1.960\sigma_X$. The 50 percent (Linear Error Probable) and 90 percent (Map Accuracy Standard) are other common linear accuracy measures and would be calculated as $0.6745\sigma_X$ and $1.645\sigma_X$, respectively. Appendix A presents a table which may be used to calculate probability indices at one percent increments.

3.2.2.7 The standard error is an accuracy formula computed as σ_X. When the normality assumption is valid, the standard error represents the interval within which 68.27 percent of the errors are within. The average error is a formula computed as $\sum |e_i| / n$. When the normality assumption is valid, the average error represents the interval within which 57.51 percent of the errors are within. The two sigma error and three sigma error are formulas which would represent 95.45 percent and 99.73 percent, respectively, under the normality assumption.

3.2.2.8. Table 1 provides conversions among the accuracy measures above. Conversions among accuracy formulas are exact.

3.2.3 Linear Non-Stationary Errors (Stochastic Processes)

3.2.3.1 An example of a linear non-stationary error in navigation may be an altimeter whose error statistics vary with actual altitude. In these cases, for each value of the independent parameter (time, distance, or altitude in this case) the error distributions can be treated as a stationary errors with the same descriptions as above. The relationships (correlations, for example, directly proportional to time or proportional to the square root of time, etc.) among the errors for different values of the independent parameter shall be specified.

3.2.4 Two-Dimensional Errors

3.2.4.1 Two-dimensional error descriptions are typically used to describe the horizontal position or velocity performance of navigation systems. Most of the discussion on one-dimensional errors is applicable. In this case the true errors are represented as pairs of values : (X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n). The true errors will in general have biases or sample mean values different from zero, designated $(\bar{X}, \bar{Y})$, where $\bar{X} = \sum X_i / n$ and $\bar{Y} = \sum Y_i / n$. These sample mean values shall be reported separately.

3.2.4.2 The unbiased or residual errors shall be the differences between the true errors and the mean values. These errors shall be designated as (e_{X_1}, e_{Y_1}), (e_{X_2}, e_{Y_2}), ..., (e_{X_n}, e_{Y_n}) where $e_{X_i} = X_i - \bar{X}$ and $e_{Y_i} = Y_i - \bar{Y}$.

3.2.4.3 The circular probability index accuracy measure for unbiased two-dimensional errors shall be the smallest R such that the probability of (e_X, e_Y) lying within the circle of radius R is greater than or equal to P, or more concisely:

$$Prob(\sqrt{e_X^2+e_Y^2} \leq R) \geq P$$

The preferred value of P for NATO applications is 0.95. The value of P of 0.5 (50 percent probability index) corresponds to the commonly used Circular Error Probable (CEP) accuracy measure.

3.2.4.4 In general, the random errors in x and y will have a non-zero sample correlation coefficient given by :

$$r = \sum e_{X_i} . e_{Y_i} \ / \ \sqrt{\sum e_{X_i}{}^2 \ . \ \sum e_{Y_i}{}^2}$$

$-1 \leq r \leq 1$. Values of r equal to +1 or -1 represent perfect correlation (y is a linear deterministic function of x), whereas a value of 0 corresponds to uncorrelated x and y errors.

3.2.4.5 When the normality assumption is valid and for large n, the probability density function for the unbiased stationary random two-dimensional errors is given by :

$$P(e_X, e_Y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-r^2}} \exp\left\{\frac{-1}{2(1-r^2)}\left[\frac{e_X^2}{\sigma_X^2}+\frac{e_Y^2}{\sigma_Y^2}-\frac{2re_Xe_Y}{\sigma_X\sigma_Y}\right]\right\}$$

This probability density function depends on the three parameters σ_X, σ_Y and r. Loci of constant density function values, called curves of constant likelihood, are generated by passing planes parallel to the e_X, e_Y plane, through the density function surface and are ellipses parallel to the e_X, e_Y plane as shown in Figure 2. That is, for specific probabilities P_1, P_2, ..., P_n concentric ellipses represent the areas within which 100 percent of the errors lie. The semi-major axis of the ellipses makes an angle θ with the positive x axis, where θ is given by:

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2r\sigma_X\sigma_Y}{\sigma_X^2-\sigma_Y^2}\right)$$

The probability, P, that the error lies within an ellipse given by:

$$\frac{e_X^2}{\sigma_X^2}+\frac{e_Y^2}{\sigma_Y^2}-\frac{2re_Xe_Y}{\sigma_X\sigma_Y} = K^2$$

is such that : $P = 1 - \exp(-K^2 / 2)$.

If e_X and e_Y were uncorrelated (r=0), then the ellipses axes would be collinear with the coordinate axis and the length of the semi-major, semi-minor axes would be 2.448 $\max(\sigma_X, \sigma_Y)$, 2.448 $\min(\sigma_X, \sigma_Y)$ for the case of P=0.95. For the case of P=0.5 the corresponding factor would be 1.1774.

3.2.4.6 Two dimensional probability indices based on elliptical areas require the specification of the parameters of the ellipse and its orientation. Except for highly skewed distributions (ratio of semi-minor axis to semi-major axis less than 0.25) it is generally preferable to use circular probability indices (see 3.2.4.3) in which case only the radius of the circle which contains a certain percentage of the unbiased errors needs to be stated. This value is independent of the orientation of the ellipse. For the circular probability indices it is convenient to perform a principal axis transformation to obtain an error distribution which has uncorrelated errors. The corresponding transformation is given by :

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} e_X \\ e_Y \end{bmatrix}$$

The resulting variances of the uncorrelated errors are:

$$\sigma_{u,v}^2 = \frac{1}{2}\left[\sigma_X^2 + \sigma_Y^2 \pm \sqrt{\sigma_X^4 + \sigma_Y^4 - 2\sigma_X^2\sigma_Y^2 + 4r^2\sigma_X^2\sigma_Y^2}\right]$$

The probability density function is now given by:

$$P(u,v) = \frac{1}{2\pi\sigma_u\sigma_v} \exp\left[-\frac{1}{2}\left(\frac{u^2}{\sigma_u^2} + \frac{v^2}{\sigma_v^2}\right)\right]$$

We note that since $R = \sqrt{e_X^2 + e_Y^2} = \sqrt{u^2 + v^2}$, the probability that (e_X, e_Y) lies within the radius R can be computed with the above uncorrelated probability density function. We designate σ_{max} as $\max(\sigma_u, \sigma_v)$, σ_{min} as $\min(\sigma_u, \sigma_v)$ and C as the aspect ratio $\sigma_{min} / \sigma_{max}$.

3.2.4.7 The probability that the unbiased two-dimensional error lies within a circles of radius $K\sigma_{max}$ is given by the following integral :

$$Prob(R < K\sigma_{max}) = \frac{2C}{\pi}\int_{\Phi=0}^{\pi}\frac{1-\exp\left[\frac{-K^2}{4C^2}\left(1+C^2-(1-C^2)\cos\Phi\right)\right]}{1+C^2-(1-C^2)\cos\Phi}d\Phi$$

The resultant probability is a function of the aspect ratio. The value of K (the factor which multiplies the σ_{max} to obtain the radius of the circle) as a function of C (the aspect ratio) is plotted in Figure 3. The following procedure would thus be utilized to obtain the 0.95 circular probability index for jointly normal distributed two dimensional navigation errors : a) remove the means from (X,Y) to form ($\tilde{X}$,$\tilde{Y}$) and separately report the means. b) compute σ_x and σ_Y and r. c) compute σ_{max}, σ_{min}, and C. d) Look up K from the 95 percent curve of Figure 3 for the corresponding value of C. e) compute the radius of the 95 percent circle as $R=K\sigma_{max}$.

3.2.4.8 Acceptable approximation formulas to compute the 50 percent (CEP) and 95 percent circular probability indices valid over the useful range of circular probability indices (0.25<C<1) are given by:

$$R_{50}=0.6142\,\sigma_{min} + 0.5632\,\sigma_{max}$$

$$R_{95}=(1.960790 + 0.004071C + 0.114276C^2 + 0.371625C^3)\,\sigma_{max}$$

For example, if σ_{max}= 16 meters and σ_{min}= 6.4 meters, then C=0.4 and the radius of the 50 percent circle would be 12.9 and the radius of the 95 percent circle would be 32.1 meters.

3.2.4.9 Conversions among the 50 percent, 90 percent, 68.3 percent, 95 percent, and 99,5 percent circular probability indices are provided in Table 2.

3.2.4.10 DRMS, 2DRMS, and Geometric Mean Error (GME) are accuracy formulas given by :

$$R(DRMS) = \sqrt{\sigma_X^2+\sigma_Y^2} = \sqrt{\sigma_{min}^2+\sigma_{max}^2}$$
$$R(2DRMS) = 2\sqrt{\sigma_X^2+\sigma_Y^2} = 2\sqrt{\sigma_{min}^2+\sigma_{max}^2}$$
$$R(GME) = 1.178\sqrt{\sigma_{min}\cdot\sigma_{max}}$$

These formula have different probabilities depending on the aspect ratio, C, as shown in Figure 4. Table 3 provides conversions among some accuracy formulas and probability indices. 2DRMS is a "safe" formula for the computation of the 95 percent radius in that it is always larger, going from about 95.4 percent (C=0) to 98.2 percent (C=1).

3.2.4.11 For the special case of $\sigma_{min} = \sigma_{max} = \sigma$, the uncorrelated errors would have circular distributions (that is concentric circles would represent areas of constant probability). Note that $\sigma_{min} = \sigma_{max}$ implies uncorrelated e_x and e_y, but $\sigma_X = \sigma_Y$ does not. The probability that the Error lies within a radius of $K\sigma$ is given by :

$$Prob(R \le K\sigma) = 1 - \exp(-K^2 / 2) \qquad (Rayleigh\ pdf)$$

The probability that the errors lie within a circle of radius 1σ is 0.3935. The 0.5 and 0.95 probability indices are given by 1.1774σ and 2.448σ, respectively. Table at Appendix 1 presents conversion factors for different probability indices at one percent increments for the special case of $\sigma_{min} = \sigma_{max}$.

3.2.4.12 Non-stationary (stochastic processes) two dimensional errors have statistics which change with an independent parameter such as mission time or distance travelled. At any particular value of the independent parameter over a large number of samples, the errors can be described as above. The relationships (correlations) among the errors at different values of the parameter must be specified. For relatively short duration aircraft inertial navigation systems, the two-dimensional horizontal position error growth with time can usually be considered linear in time relative to mission initialization. In this case, the non-stationary horizontal position error growth rate will be expressed as the radius of the circles per unit time (expressed in units of length per unit time, e.g., nmi/hour (knots)) within which 95 percent of a large number of observations will be contained.

3.2.5 Three Dimensional Stationary Random Errors

3.2.5.1 The probability index accuracy measure for unbiased three-dimensional errors shall be the smallest R such that the probability of (e_X, e_Y, e_Z) lying within the sphere of radius R is greater than or equal to P, or more concisely :

$$Prob(\sqrt{e_X^2+e_Y^2+e_Z^2} \leq R) \geq P$$

The preferred value of P for NATO applications is 0.95. The value of P of 0.5 (50 percent probability index) corresponds to the commonly used Spherical Error Probable (SEP) accuracy measure.

3.2.5.2 For most of the discussion, direct extensions from the one and two dimensional cases may be drawn. The unbiased stationary normal random errors (e_X, e_Y, e_Z) with standard deviations σ_X, σ_Y, σ_Z and sample cross correlation coefficients r_{XY}, r_{XZ}, and r_{YZ} have a joint probability density function :

$$P(\vec{e}) = \frac{1}{(2\pi)^{3/2}\,|P|^{1/2}} \exp\left(-\frac{1}{2}\vec{e}^{\,T}[P]^{-1}\vec{e}\right)$$

where, [P], the covariance matrix is given by :

$$[P] = \begin{bmatrix} \sigma_X^2 & r_{XY}\sigma_X\sigma_Y & r_{XZ}\sigma_X\sigma_Z \\ r_{XY}\sigma_X\sigma_Y & \sigma_Y^2 & r_{YZ}\sigma_Y\sigma_Z \\ r_{XZ}\sigma_X\sigma_Z & r_{YZ}\sigma_Y\sigma_Z & \sigma_Z^2 \end{bmatrix}$$

3.2.5.3 Surfaces of constant probability density are called surfaces of constant likelihood. They are ellipsoids with principal axes not generally aligned with the navigation coordinate axes.

3.2.5.4 It is convenient to perform a linear orthogonal principal axis (eigenvector) transformation such that the constant density ellipsoids of the new variables become aligned with the axes of the new variables. The uncorrelated errors (u, v, w) have a joint probability density function given by :

$$P(u,v,w) = \frac{1}{(2\pi)^{3/2}\sigma_u\sigma_v\sigma_w} \exp\left[-\frac{1}{2}\left(\frac{u^2}{\sigma_u^2} + \frac{v^2}{\sigma_v^2} + \frac{w^2}{\sigma_w^2}\right)\right]$$

We designate the minimum, median, and maximum values of $(\sigma_u, \sigma_v, \sigma_w)$ as : $\sigma_{min}, \sigma_{mid}, \sigma_{max}$.

The non-aligned ellipsoid governed by the equation:

$$\vec{e}^{\,T}[P]^{-1}\vec{e} = w^2$$

has the probability that (e_X, e_Y, e_Z) lie within the ellipsoid such that :

$$Prob\left\{(e_X, e_Y, e_Z) \text{ within ellipsoid}\right\} = \sqrt{\frac{2}{\pi}} \int_0^w t^2 e^{-t^2/2}\, dt$$

where t is a dummy integration variable. The probability is 50 percent that (e_X, e_Y, e_Z) lies within an ellipsoid with semi-major, semi-mid axis, and semi-minor axis given by $1.538\,\sigma_{max}$, $1.538\,\sigma_{mid}$ and $1.538\,\sigma_{min}$. The probability is 95 percent that (e_X, e_Y, e_Z) lies within an ellipsoid with semi-major, semi-mid axis, and semi-minor axis given by $2.7955\,\sigma_{max}$, $2.7955\,\sigma_{mid}$ and $2.7955\,\sigma_{min}$.

3.2.5.5 Ellipsoidal probability indices as above require the specification of the semi-minor, semi-mid, and semi-major axis of the ellipse as well as the angles that the principal axes of the ellipse make with the navigation coordinate axes. In general it is preferred to use a spherical probability index (paragraph 3.2.5.1) such that only the radius of the sphere need be specified. Ellipsoidal probability indices shall be used for normally distributed three dimensional distributions if $\sigma_{min}/\sigma_{max}$ is less than 0.25.

3.2.5.6 Graphs of three dimensional circular probability indices are provided in Figure 5 with $C_1=\sigma_{min}/\sigma_{max}$ and $C_2=\sigma_{mid}/\sigma_{max}$ as parameters. Thus the following procedure can be used to determine the three dimensional circular probability indices : a) remove the means from (X, Y, Z) to form $(\bar{X},\bar{Y},\bar{Z})$ and separately report the means. b) compute σ_X, σ_Y, σ_Z, r_{XY}, r_{XZ}, and r_{YZ}. c) diagonalize the covariance matrix to obtain σ_{max}, σ_{mid}, σ_{min}, d) compute C_1 and C_2. e) From Figure 5 linearly interpolate between the two curves closest to the value of C_2 for the 95 percent probability to obtain the value of K at the value for C_1. f) compute the radius of the 95 percent circle as $R = K\sigma_{max}$.

3.2.5.7 Acceptable approximation formulas to compute the 50 percent (SEP) and 95 percent spherical probability indices

valid over the useful range of spherical indices are :

$$R_{50} = 0.5127\,(\sigma_{min}+\sigma_{mid}+\sigma_{max})$$
$$R_{95} = \sigma_{max}\,(2.0248-0.0776\,C_1+0.4056\,C_1^2-0.3511\,C_2+0.7793\,C_2^2)$$

3.2.5.7.1 Conversions among the 50 percent, 68.3 percent, and 95 percent spherical indices are given in Table 4.

3.2.5.8 The Mean Radial Spherical Error (MSRE) is an accuracy measure formula given by : $\sqrt{\sigma_{min}{}^2 + \sigma_{mid}{}^2 + \sigma_{max}{}^2}$.

For $\sigma_{min} = \sigma_{mid} = \sigma_{max}$, 60.82 percent of the errors lie within the MRSE. Figure 6 gives the probabilities associated with MRSE.

3.2.5.9 For $\sigma_{min} = \sigma_{mid} = \sigma_{max} = \sigma$, the uncorrelated (e_X, e_Y, e_Z) have a spherical error distribution given by:

$$P(e_X, e_Y, e_Z) = \frac{1}{(2\pi)^{3/2}\sigma^3} \exp(-s^2 / 2\sigma^2) \qquad \text{(Maxwell pdf)}$$

$$\text{where } s = \sqrt{e_X^2 + e_Y^2 + e_Z^2}$$

The probability that the error lies within a sphere of radius K is given by:

$$Prob(\sqrt{e_X^2 + e_Y^2 + e_Z^2} < K\sigma) = \left[\frac{-2}{\sqrt{\pi}}\right] \frac{K}{\sqrt{2}} e^{-K^2/2} + \text{erf}\left[\frac{K}{\sqrt{2}}\right]$$

The probability is 50 percent that the errors lie within a radius of 1.538σ. The probability is 95 percent that the error lies within 2.7955σ. Conversions among probability indices for the spherical distribution are tabulated in Appendix 1 at one percent increments.

3.2.5.9.1. An acceptable approximation to the above which does not involve computation of the error function is :

$$Prob\left(\sqrt{e_X^2+e_Y^2+e_Z^2} \le K\sigma\right) = \sqrt{\frac{2}{\pi}}\left[1.253 - K\exp(-K^2/2) - \frac{\exp(-K^2/2)}{K + 0.8e^{-0.4K}}\right]$$

3.2.6. Non-stationary three dimensional errors are not usually treated in navigation (typically the errors would be partitioned into two-dimensional and one-dimensional error).

3.2.7 The 95 percent probability indices for linear, two-dimensional and three-dimensional are statistics computed from the random error samples (of course the statistics could also be derived from a covariance analysis), The greater the number of observations we have the higher the probability that the statistic is within a specific range about the true value for the underlying population. These specific ranges are confidence intervals and can be expressed as a percentage of the true value, and are a function of the number of observations, (size of the confidence interval goes down as number of observations increases) and the probability that the statistic is within the confidence interval (size of the confidence interval goes up as the probability goes up). Thus we may say, for example, that the computed 95 percent probability index accuracy measure is within plus or minus 30 percent of the true value of the underlying population with a probability of 0,9 for the case of 16 measurements. Figure 6a presents sample size requirements for 100ϵ percent confidence intervals at probability levels 0.90, 0.95, and 0.99 for one-dimensional probability indices. Figure 6b presents sample size requirements for 100ϵ percent confidence intervals at the 0.95 probability level for two dimensional probability indices at C=1 and C=0.

4.0 Postscript

4.1 The above data processing procedures are essentially a classical treatment where assumptions such as normality and ergodicity are made. As eloquently stated by Mertikas et al:

"... rigid, stereotyped formulas leading to an unquestionable and unequivocal conclusions ..." may cause false assurances and sad surprises. When the above assumptions are in question and/or there are indications of a mismatch between sampled and target populations of navigation errors, alternate approaches should be pursued.

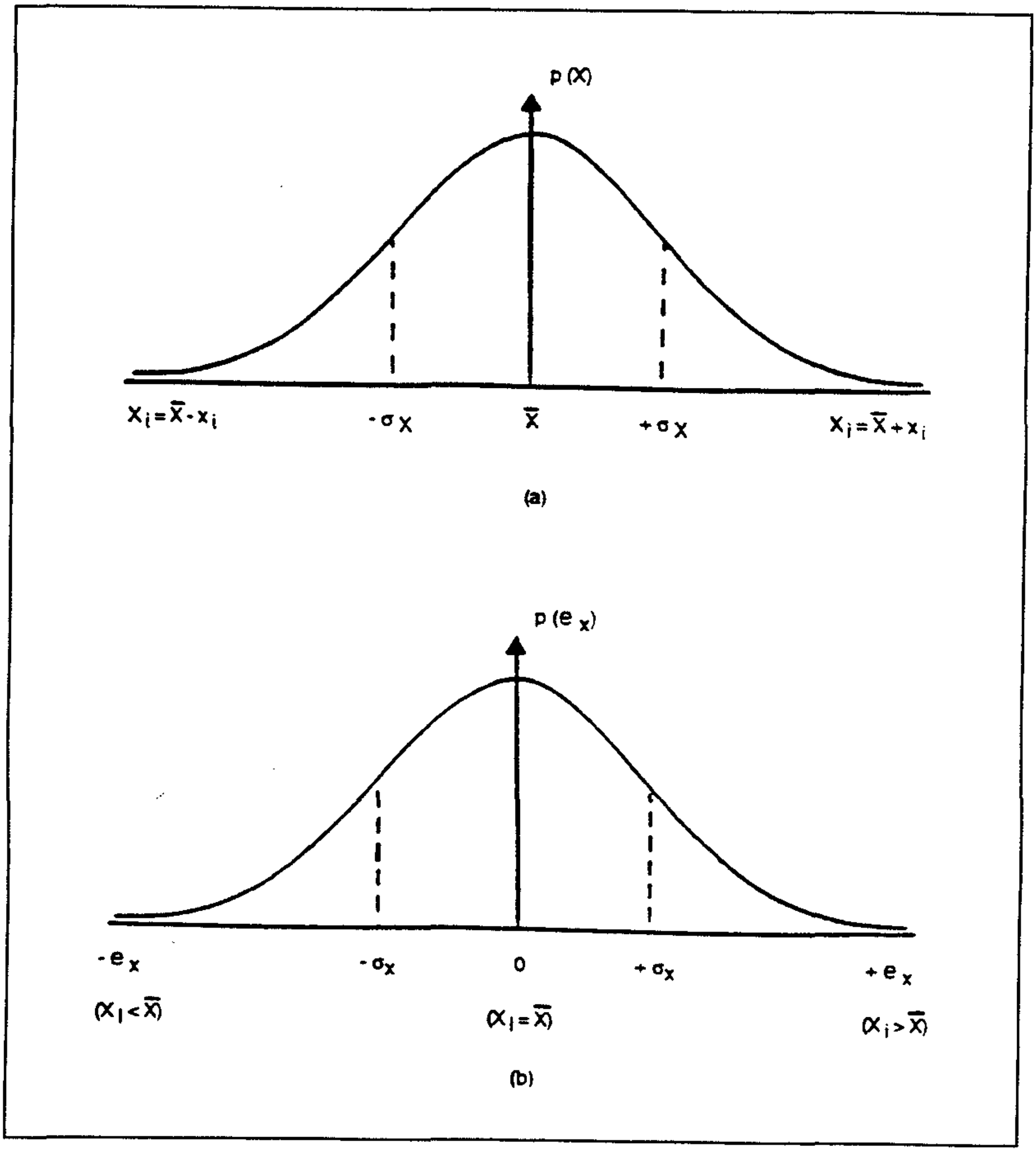

Figure 1 - Assumed Normal Probability Density Functions for Sample True Linear Random Errors (a) and Unbiased Errors (b)

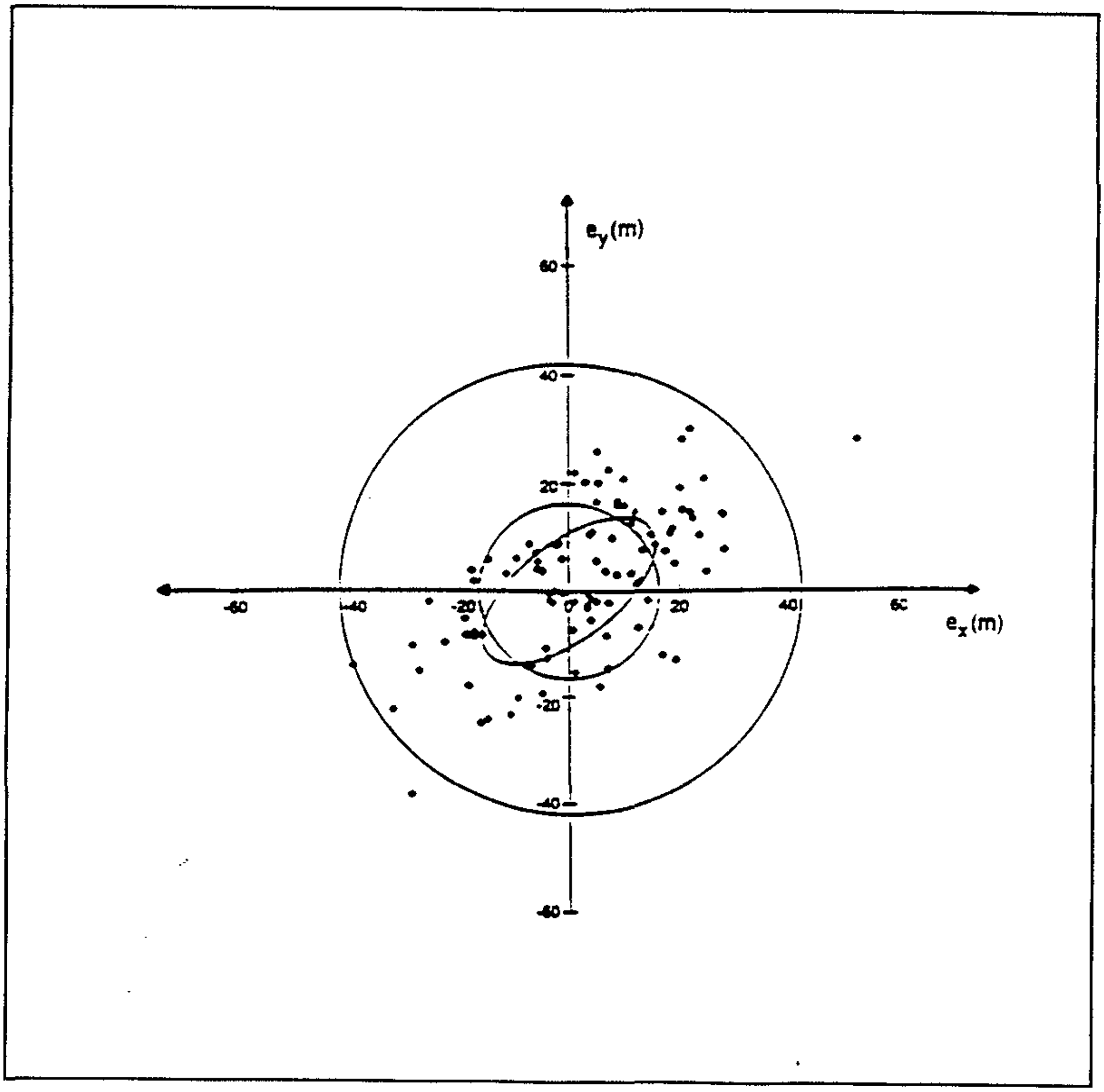

Figure 2 - A sample of 100 normally distributed horizontal position fix errors. These artificially generated errors have standard deviations in latitude and longitude of 15.8m and 13.6m respectively, with a correlation of 0.64. Shown are the error ellipse, the circle with 2DRMS (41.8m) radius, and the circle with a radius equal to the CEP (16.4m) corresponding to this sample.

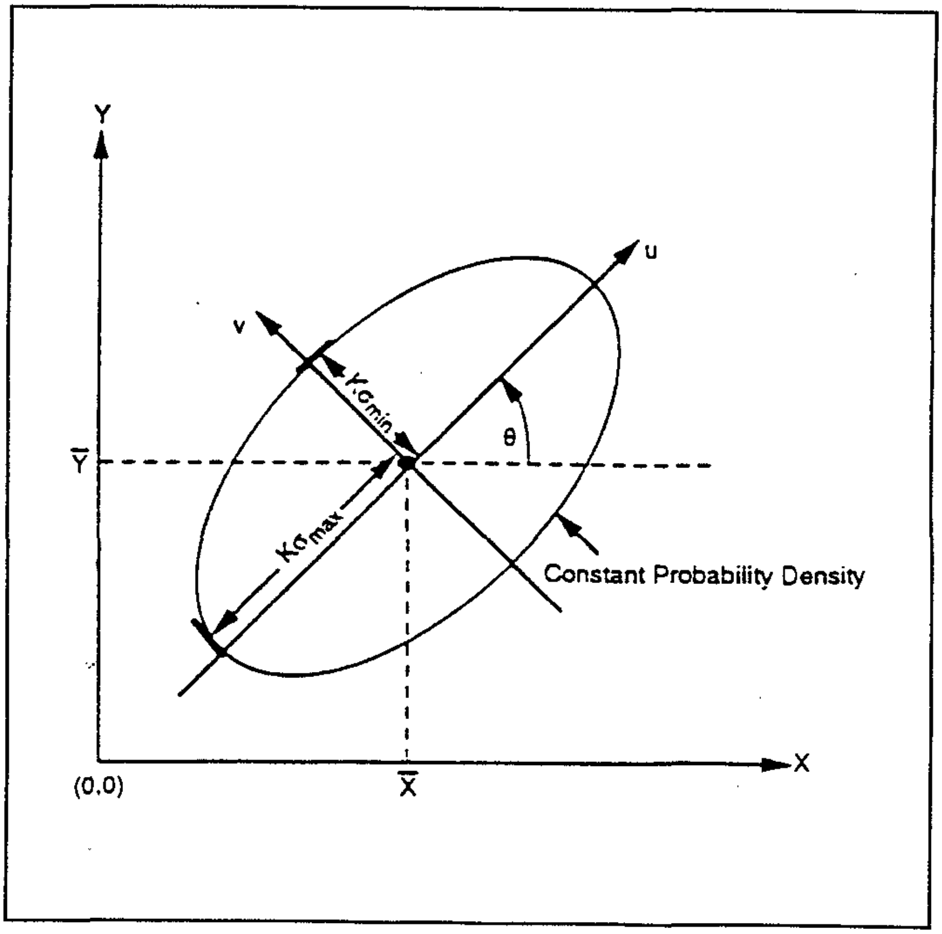

Figure 2 - Prob{ (e_X, e_Y) within Ellipse} = $1 - e^{-\frac{K^2}{2}}$

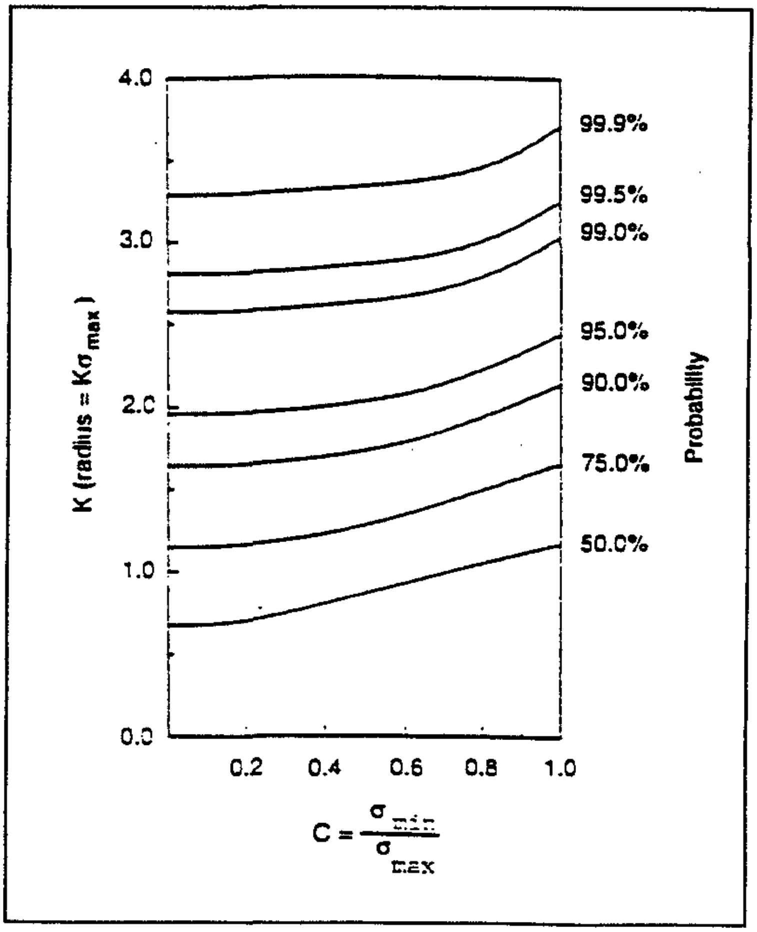
4.0
3.0
2.0
1.0
0.0
K (radius = Kσ max)
0.2
0.4
0.6
0.8
1.0
C = σ min / σ max
99.9%
99.5%
99.0%
95.0%
90.0%
75.0%
50.0%
Probability

Figure 3

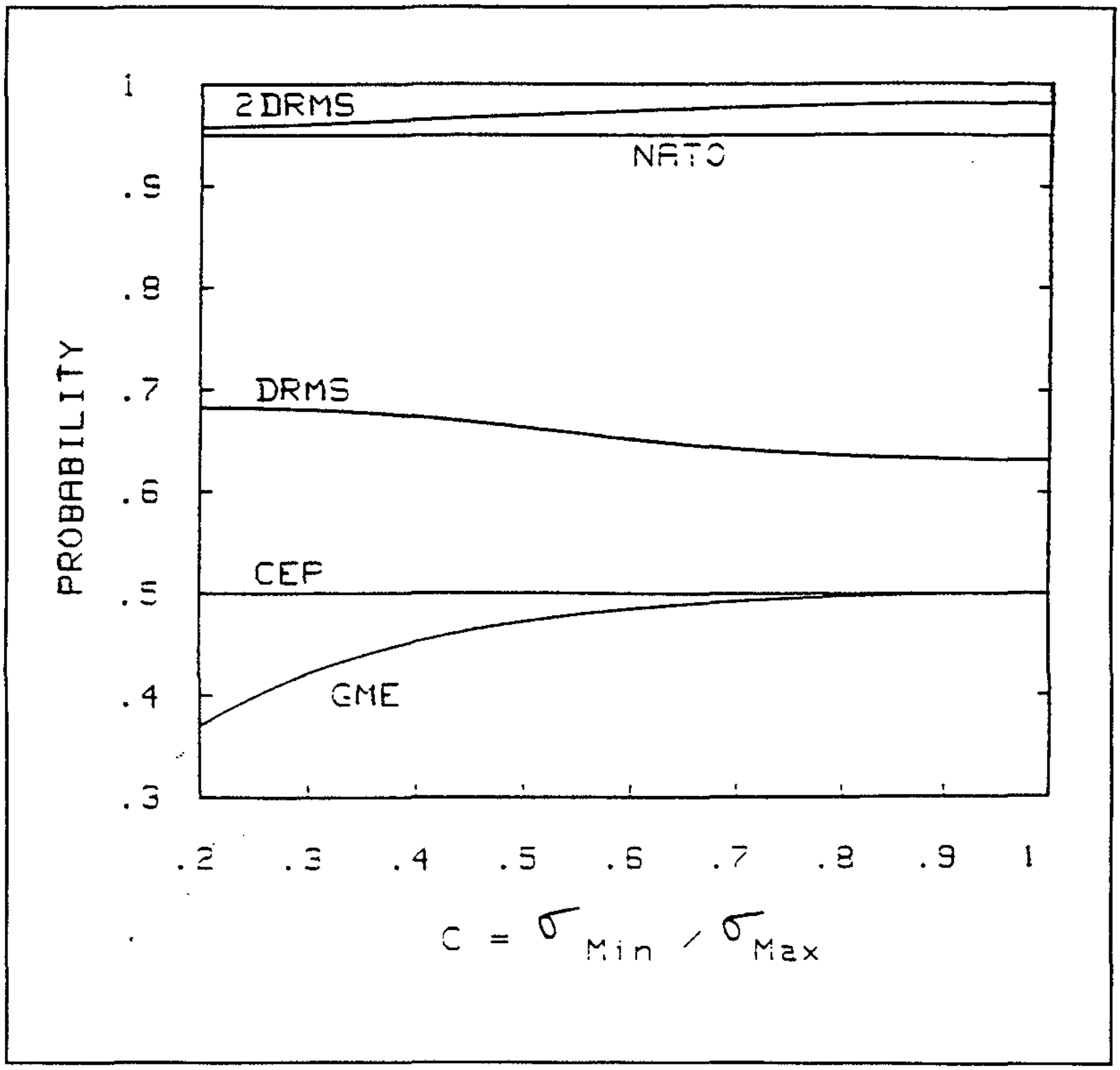
1
.9
.8
.7
.6
.5
.4
.3
PROBABILITY
2DRMS
NATO
DRMS
CEP
GME
.2
.3
.4
.5
.6
.7
.8
.9
1
C = σ Min / σ Max

Figure 4

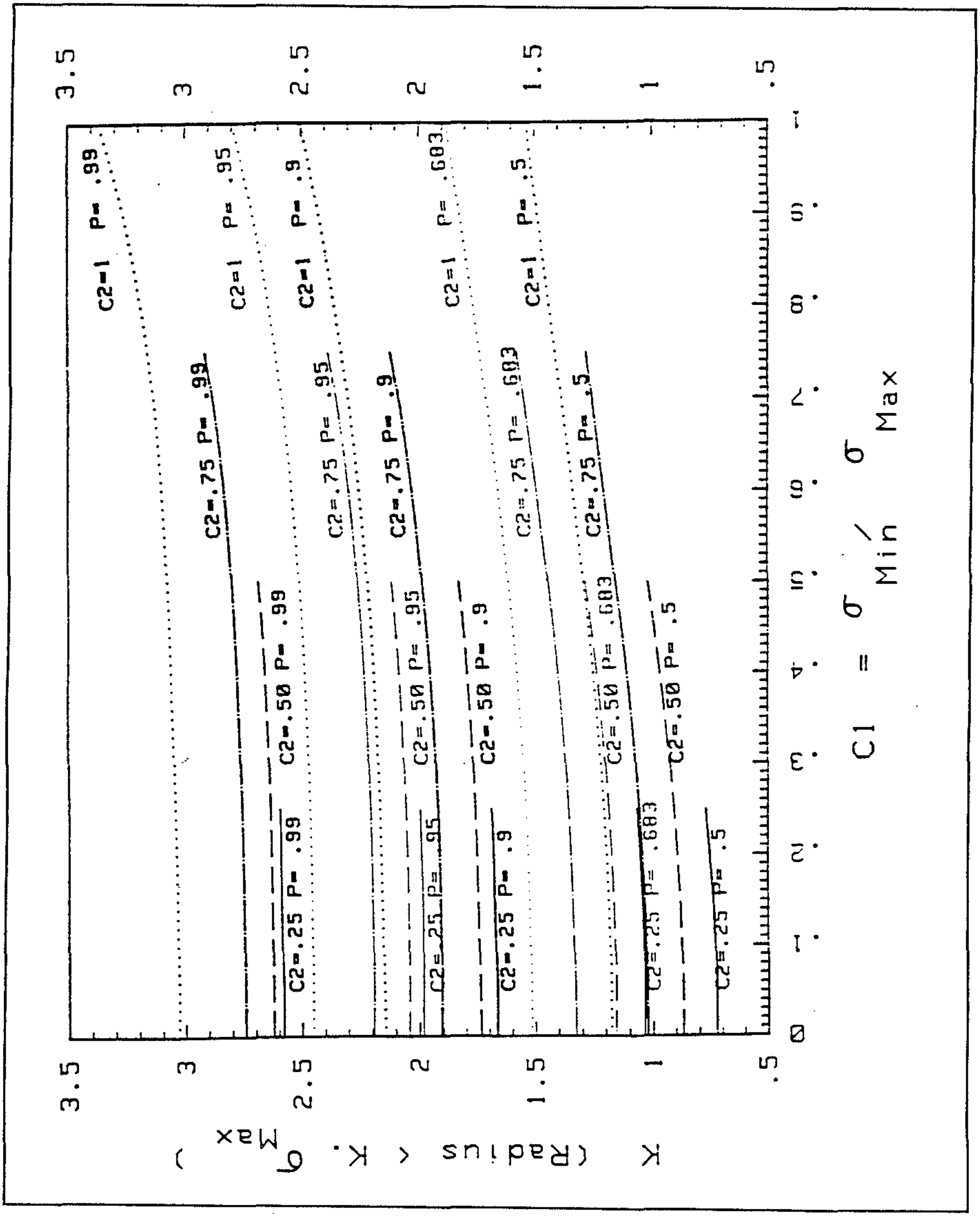

Figure 5 - Three Dimensional Probability Indices

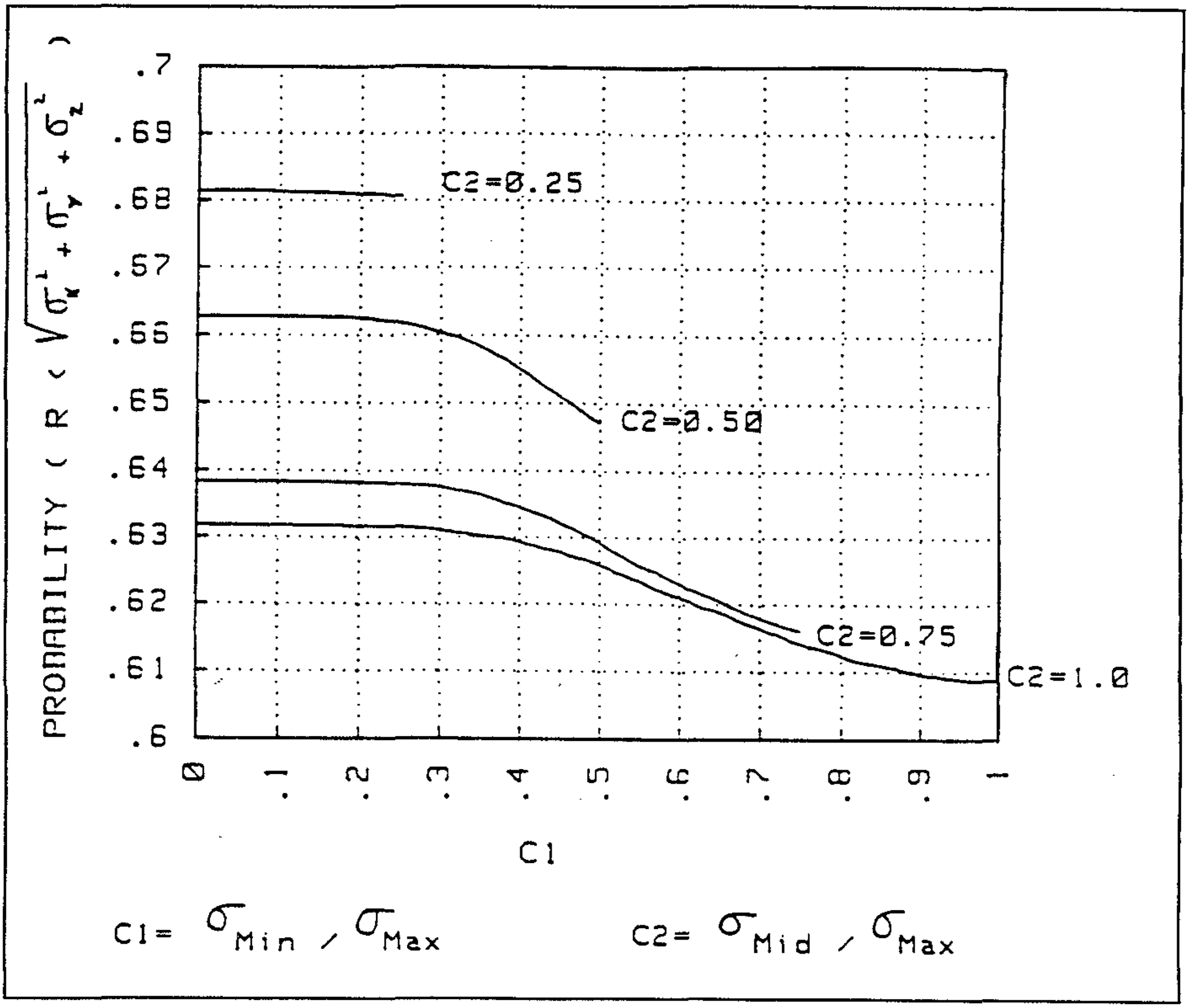

Figure 6 - Three Dimensional MSRE Probability

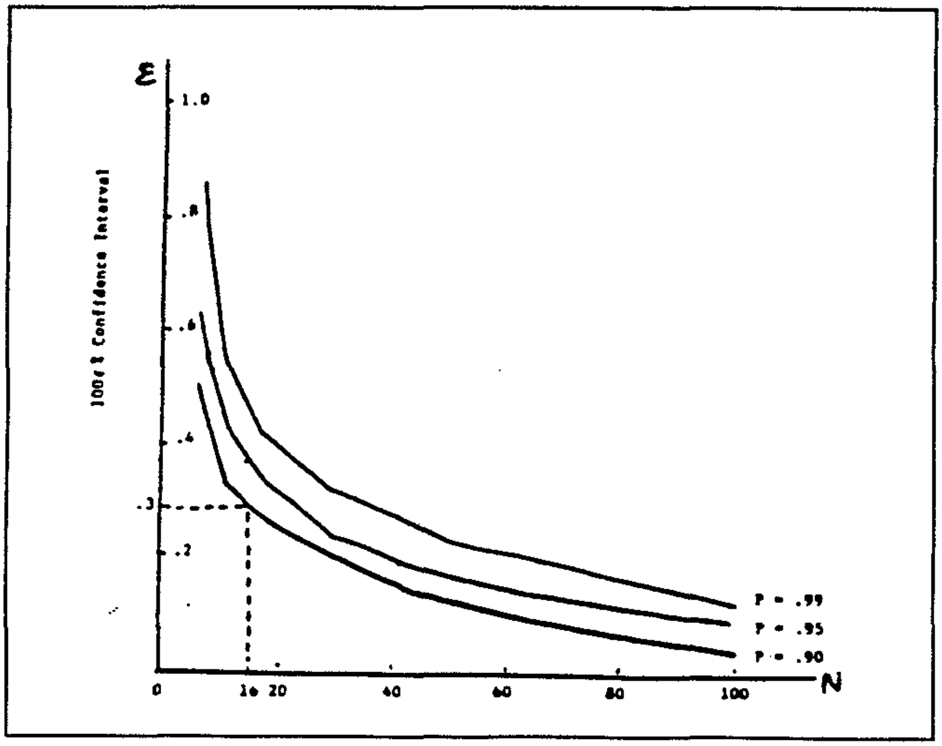

Figure 6a - Sample Size Requirement for 100ε Percent Confidence Intervals for One Dimension

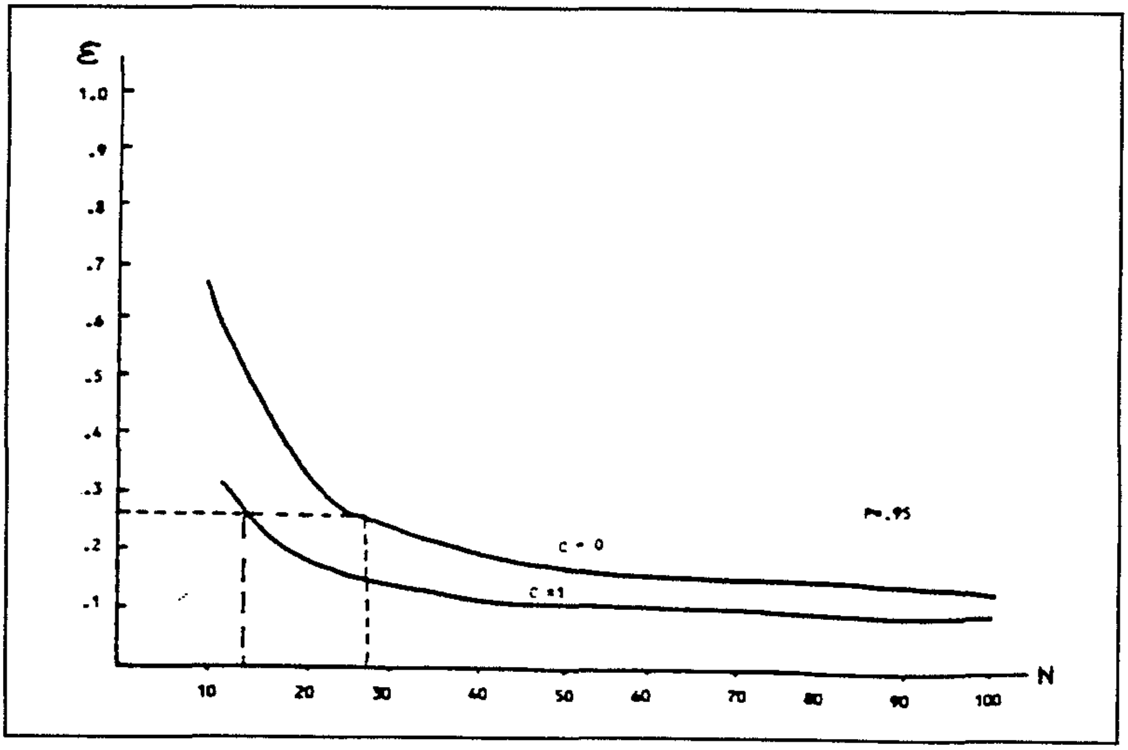

Figure 6b - Sample Size Requirements for 100ϵ Percent Confidence Intervals for Two Dimensions

FROM \ TO	50% (LEP)	90% (MAS)	95% (NATO)	One Sigma	Two Sigma	Three Sigma	Average Error
50% (LEP)	1	2.4387	2.9059	1.4826	2.9652	4.4478	1.1829
90% (MAS)	0.4100	1	1.1910	0.6079	1.2158	1.8237	0.4851
95% (NATO)	0.3441	0.8396	1	0.5102	1.0204	1.5306	0.4071
One Sigma	0.6745	1.6450	1.9600	1	2	3	N/A
Two Sigma	0.3372	0.8225	0.9800	0.5000	1	1.5	N/A
Three Sigma	0.2248	0.5483	0.0653	0.3333	0.6667	1	N/A
Average Error	0.8454	2.0614	2.4564	N/A	N/A	N/A	1

To Convert from Row to Column, Multiply by Table Values

Table 1 - Summary of One Dimensional Probability Indices and Accuracy Formulas Conversion

FROM \ TO		50%					68.3%					90%					95%					99.5%				
		C=0.2	C=0.4	C=0.6	C=0.8	C=1.0	C=0.2	C=0.4	C=0.6	C=0.8	C=1.0	C=0.2	C=0.4	C=0.6	C=0.8	C=1.0	C=0.2	C=0.4	C=0.6	C=0.8	C=1.0	C=0.2	C=0.4	C=0.6	C=0.8	C=1.0
50%	C=0.2	1.000					1.447					2.348					2.792					3.990				
	C=0.4		1.000					1.355					2.104					2.483					3.519			
	C=0.6			1.000					1.307					1.919					2.230					3.101		
	C=0.8				1.000					1.291					1.841					2.109					2.849	
	C=1.0					1.000					1.287					1.823					2.080					2.774
68.3%	C=0.2	0.691					1.000					1.623					1.929					2.758				
	C=0.4		0.738					1.000					1.553					1.832					2.597			
	C=0.6			0.765					1.000					1.468					1.706					2.372		
	C=0.8				0.775					1.000					1.426					1.634					2.206	
	C=1.0					0.777					1.000					1.416					1.615					2.155
90%	C=0.2	0.426					0.616					1.000					1.189					1.699				
	C=0.4		0.475					0.644					1.000					1.180					1.673			
	C=0.6			0.521					0.681					1.000					1.162					1.616		
	C=0.8				0.543					0.701					1.000					1.146					1.547	
	C=1.0					0.549					0.706					1.000					1.141					1.522
95%	C=0.2	0.358					0.518					0.841					1.000					1.429				
	C=0.4		0.403					0.546					0.847					1.000					1.417			
	C=0.6			0.448					0.586					0.861					1.000					1.391		
	C=0.8				0.474					0.612					0.873					1.000					1.350	
	C=1.0					0.481					0.619					0.877					1.000					1.334
99.5%	C=0.2	0.251					0.363					0.588					0.700					1.000				
	C=0.4		0.284					0.385					0.598					0.705					1.000			
	C=0.6			0.322					0.422					0.619					0.719					1.000		
	C=0.8				0.351					0.453					0.646					0.740					1.000	
	C=1.0					0.360					0.464					0.657					0.750					1.000

To Convert from Row to Column, Multiply by Table Values

Table 2 - Conversion factors for two dimensional jointly Normal Accuracy Measures. Valid for the same value of $\sigma_{min}/\sigma_{max}$ only.

FROM \ TO		DRMS					2DRMS					CEP					68.3%					NATO				
		C=0.2	C=0.4	C=0.6	C=0.8	C=1.0	C=0.2	C=0.4	C=0.6	C=0.8	C=1.0	C=0.2	C=0.4	C=0.6	C=0.8	C=1.0	C=0.2	C=0.4	C=0.6	C=0.8	C=1.0	C=0.2	C=0.4	C=0.6	C=0.8	C=1.0
DRMS	C=0.2	1.000					2.000					0.692					1.002					1.932				
	C=0.4		1.000					2.000					0.750					1.016					1.862			
	C=0.6			1.000					2.000					0.801					1.047					1.785		
	C=0.8				1.000					2.000					0.826					1.067					1.742	
	C=1.0					1.000					2.000					0.833					1.072					1.732
2DRMS	C=0.2	0.500					1.000					0.346					0.501					0.966				
	C=0.4		0.500					1.000					0.375					0.508					0.931			
	C=0.6			0.500					1.000					0.400					0.523					0.893		
	C=0.8				0.500					1.000					0.413					0.533					0.871	
	C=1.0					0.500					1.000					0.416					0.536					0.866
CEP	C=0.2	1.445					2.889					1.000					1.447					2.792				
	C=0.4		1.333					2.666					1.000					1.355					2.483			
	C=0.6			1.249					2.498					1.000					1.307					2.230		
	C=0.8				1.211					2.421					1.000					1.291					2.109	
	C=1.0					1.201					2.402					1.000					1.287					2.000
68.3%	C=0.2	0.998					1.997					0.691					1.000					1.929				
	C=0.4		0.984					1.968					0.738					1.000					1.832			
	C=0.6			0.955					1.911					0.765					1.000					1.706		
	C=0.8				0.938					1.875					0.775					1.000					1.634	
	C=1.0					0.933					1.666					0.777					1.000					1.615
NATO	C=0.2	0.517					1.035					0.358					0.518					1.000				
	C=0.4		0.537					1.074					0.403					0.546					1.000			
	C=0.6			0.560					1.120					0.448					0.586					1.000		
	C=0.8				0.574					1.148					0.474					0.612					1.000	
	C=1.0					0.577					1.155					0.481					0.619					1.000

To Convert from Row to Column, Multiply by Table Values

Table 3 - Conversion factors for two dimensional jointly Normal Accuracy Measures. Valid for the same value of $\sigma_{min}/\sigma_{max}$ only.

C_{mid}=0.25 C_{min}=0.25

TO / FROM	50%	68.3%	95%	99%	MSRE
50%	1.000	1.380	2.580	3.361	0.882
68.3%	0.725	1.000	1.870	2.437	0.639
95%	0.388	0.535	1.000	1.303	0.342
99%	0.297	0.410	0.768	1.000	0.262
MSRE	1.134	1.564	2.925	3.811	1.000

C_{mid}=0.50 C_{min}=0.25

TO / FROM	50%	68.3%	95%	99%	MSRE
50%	1.000	1.303	2.261	2.904	0.729
68.3%	0.767	1.000	1.735	2.228	0.560
95%	0.442	0.576	1.000	1.284	0.323
99%	0.344	0.449	0.779	1.000	0.251
MSRE	1.371	1.787	3.101	3.982	1.000

C_{mid}=0.50 C_{min}=0.50

TO / FROM	50%	68.3%	95%	99%	MSRE
50%	1.000	1.256	2.068	2.633	0.634
68.3%	0.796	1.000	1.646	2.095	0.505
95%	0.484	0.608	1.000	1.273	0.307
99%	0.380	0.477	0.785	1.000	0.241
MSRE	1.577	1.982	3.262	4.153	1.000

C_{mid}=0.75 C_{min}=0.25

TO / FROM	50%	68.3%	95%	99%	MSRE
50%	1.000	1.279	2.080	2.606	0.603
68.3%	0.782	1.000	1.626	2.037	0.471
95%	0.481	0.615	1.000	1.253	0.290
99%	0.384	0.491	0.798	1.000	0.231
MSRE	1.659	2.122	3.451	4.324	1.000

To Convert from Row to Column, Multiply by Table Values

Table 4 - Conversion Factors for Three Dimensional Jointly Normal Accuracy Measures

C_{mid}=0.75 C_{min}=0.50

TO / FROM	50%	68.3%	95%	99%	MSRE
50%	1.000	1.245	1.951	2.424	0.545
68.3%	0.803	1.000	1.568	1.948	0.438
95%	0.513	0.638	1.000	1.243	0.279
99%	0.412	0.513	0.805	1.000	0.225
MSRE	1.835	2.284	3.581	4.449	1.000

C_{mid}=0.75 C_{min}=0.75

TO / FROM	50%	68.3%	95%	99%	MSRE
50%	1.000	1.230	1.859	2.277	0.482
68.3%	0.813	1.000	1.511	1.850	0.392
95%	0.538	0.662	1.000	1.224	0.259
99%	0.439	0.540	0.817	1.000	0.212
MSRE	2.076	2.554	3.860	4.726	1.000

C_{mid}=1.00 C_{min}=0.25

TO / FROM	50%	68.3%	95%	99%	MSRE
50%	1.000	1.275	2.040	2.525	0.524
68.3%	0.784	1.000	1.600	1.981	0.411
95%	0.490	0.625	1.000	1.238	0.257
99%	0.396	0.505	0.808	1.000	0.207
MSRE	1.909	2.434	3.894	4.821	1.000

C_{mid}=1.00 C_{min}=0.50

TO / FROM	50%	68.3%	95%	99%	MSRE
50%	1.000	1.247	1.942	2.388	0.485
68.3%	0.802	1.000	1.557	1.914	0.389
95%	0.515	0.642	1.000	1.230	0.250
99%	0.419	0.522	0.813	1.000	0.203
MSRE	2.061	2.571	4.002	4.921	1.000

To Convert from Row to Column, Multiply by Table Values

Table 4 (Cont'd) - Conversion Factors for Three Dimensional Jointly Normal Accuracy Measures

C_{mid}=1.00 C_{min}=0.75

TO / FROM	50%	68.3%	95%	99%	MSRE
50%	1.000	1.228	1.847	2.243	0.436
68.3%	0.814	1.000	1.504	1.826	0.355
95%	0.541	0.665	1.000	1.214	0.236
99%	0.446	0.548	0.824	1.000	0.194
MSRE	2.296	2.819	4.241	5.148	1.000

C_{mid}=1.00 C_{min}=1.00

TO / FROM	50%	68.3%	95%	99%	MSRE
50%	1.000	1.224	1.820	2.192	0.396
68.3%	0.817	1.000	1.487	1.791	0.324
95%	0.550	0.673	1.000	1.205	0.218
99%	0.456	0.558	0.830	1.000	0.181
MSRE	2.524	3.089	4.592	5.532	1.000

To Convert from Row to Column, Multiply by Table Values

Table 4 (Cont'd) - Conversion Factors for Three Dimensional Jointly Normal Accuracy Measures

PERCENTAGE PROBABILITY FOR STANDARD ERROR INCREMENTS

The following table presents the increments of linear (σ_X), circular (σ_C) and spherical (σ_S) standard errors for intervals of one percent probability. Percentage levels corresponding to precision indexes are underlined. The conversions for two and three dimensions are strictly valid only when $\sigma_{max}=\sigma_{min}$. See paragraphs 3.2.4.11 and 3.2.5.6.

Factors for converting the error at one percentage probability to another within the same distribution are derived by dividing the standard error increment of the new percentage probability by the standard error increment of the given percentage probability. An example is the conversion from the NATO accuracy standard (95%) to the circular probable error (50%).

$$\text{CEP} = 1.1774\ \sigma_C$$

$$R_{95}\ (\text{NATO}) = 2.4477\ \sigma_C$$

$$\text{CEP} = \frac{1.1774}{2.4477}\ \text{NATO}$$

$$\text{CEP} = 0.4810\ \text{NATO}$$

%	σ_X	σ_C	σ_S
00	0.0000	0.0000	0.0000
01	0.0125	0.1418	0.3369
02	0.0251	0.2010	0.4299
03	0.0376	0.2468	0.4951
04	0.0502	0.2857	0.5479
05	0.0627	0.3203	0.5932
06	0.0753	0.3518	0.6334
07	0.0878	0.3810	0.6699
08	0.1004	0.4084	0.7035
09	0.1130	0.4343	0.7349
10	0.1257	0.4590	0.7644
11	0.1383	0.4828	0.7924

%	σ_x	σ_c	σ_s
12	0.1510	0.5056	0.8192
13	0.1637	0.5278	0.6447
14	0.1764	0.5492	0.8694
15	0.1891	0.5701	0.8932
16	0.2019	0.5905	0.9162
17	0.2147	0.6105	0.9386
18	0.2275	0.6300	0.9605
19	0.2404	0.6492	0.9818
19.9			1.0000
20	0.2533	0.6680	1.0026
21	0.2663	0.6866	1.0230
22	0.2793	0.7049	1.0430
23	0.2924	0.7230	1.0627
24	0.3055	0.7409	1.0821
25	0.3186	0.7585	1.1012
26	0.3319	0.7760	1.1200
27	0.3451	0.7934	1.1386
28	0.3585	0.8106	1.1570
29	0.3719	0.8276	1.1751
30	0.3853	0.8446	1.1932
31	0.3989	0.8615	1.2110
32	0.4125	0.8783	1.2288
33	0.4261	0.8950	1.2464
34	0.4399	0.9116	1.2638
35	0.4538	0.9282	1.2812
36	0.4677	0.9448	1.2985
37	0.4817	0.9613	1.3158
38	0.4959	0.9778	1.3330
39	0.5101	0.9943	1.3501
39.35		1.0000	
40	0.5244	1.0108	1.3672
41	0.5388	1.0273	1.3842
42	0.5534	1.0438	1.4013
43	0.5681	1.0603	1.4183

%	σ_x	σ_c	σ_s
44	0.5828	1.0769	1.4354
45	0.5978	1.0935	1.4524
46	0.6128	1.1101	1.4695
47	0.6280	1.1268	1.4866
48	0.6433	1.1436	1.5037
49	0.6588	1.1605	1.5209
50	0.6745	1.1774	1.5382
51	0.6903	1.1944	1.5555
52	0.7063	1.2116	1.5729
53	0.7225	1.2288	1.5904
54	0.7388	1.2462	1.6080
55	0.7554	1.2637	1.6257
56	0.7722	1.2814	1.6436
57	0.7892	1.2992	1.6616
57.51	0.7979		
58	0.8064	1.3172	1.6797
59	0.8239	1.3354	1.6980
60	0.8416	1.3537	1.7164
60.82			1.7321
61	0.8596	1.3723	1.7351
62	0.8779	1.3911	1.7540
63	0.8965	1.4101	1.7730
63.21		1.4142	
64	0.9154	1.4294	1.7924
65	0.9346	1.4490	1.8119
66	0.9542	1.4689	1.8318
67	0.9741	1.4891	1.8519
68	0.9945	1.5096	1.8724
68.27	1.0000		
69	1.0152	1.5305	1.8932
70	1.0364	1.5518	1.9144
71	1.0581	1.5735	1.9360
72	1.0803	1.5956	1.9580
73	1.1031	1.6182	1.9804

%	σ_x	σ_c	σ_s
74	1.1264	1.6414	2.0034
75	1.1503	1.6651	2.0269
76	1.1750	1.6894	2.0510
77	1.2004	1.7145	2.0757
78	1.2265	1.7402	2.1012
79	1.2536	1.7667	2.1274
80	1.2816	1.7941	2.1544
81	1.3106	1.8225	2.1825
82	1.3408	1.8519	2.2114
83	1.3722	1.8825	2.2416
84	1.4051	1.9145	2.2730
85	1.4395	1.9479	2.3059
86	1.4758	1.9830	2.3404
87	1.5141	2.0200	2.3767
88	1.5548	2.0593	2.4153
89	1.5982	2.1011	2.4563
90	1.6449	2.1460	2.5003
91	1.6954	2.1945	2.5478
92	1.7507	2.2475	2.5998
93	1.8119	2.3062	2.6571
94	1.8808	2.3721	2.7216
95	1.9600	2.4477	2.7955
96	2.0537	2.5373	2.8829
97	2.1701	2.6482	2.9912
98	2.3263	2.7971	3.1365
99	2.5758	3.0349	3.3683
99.73	3.0000		
99.78		3.5000	
99.89			4.0000
99.9	3.2905	3.7129	4.0345
99.99	3.8905	4.2919	4.6094

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